

# The Axiomatization of Overriding and Update<sup>☆</sup>

## Online Appendix: Human-Readable Proofs

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### Abstract

This document contains human-readable proofs for the derived laws in Figure 3 of the article published under the same title in the Journal of Applied Logic. The proofs are mostly based on the ones provided by ISABELLE.

In the following, we provide equational proofs for all laws in Figure 3. Lemmas are tagged with the number of the law and with a letter to indicate the order of proofs: in each proof, only laws with an earlier letter may be used. If a rule is only applied to a part of the formula, the part that is going to be replaced is red and underlined, and the part that has just been replaced is blue and overlined. The proofs are mostly based on the ones provided by ISABELLE; they differ mainly if the automatic proof for one law contains (most of) the proof for another.

We assume left-associativity and the usual priorities: @ binds strongest and  $\triangleright$  binds weakest.

The axioms are:

$$x \triangleright x = x \quad (\text{Ax 1})$$

$$x - x = \emptyset \quad (\text{Ax 2})$$

$$x - y \triangleright y = y \triangleright x \quad (\text{Ax 3})$$

$$x - (y - z) = x - y \triangleright x @ z \quad (\text{Ax 4})$$

$$(x \triangleright y) - z = x - z \triangleright y - z \quad (\text{Ax 5})$$

The operator @ is defined as an abbreviation:

$$x @ y = x - (x - y) \quad (@)$$

### 1. Proof based on Axioms 1–3

$$\begin{aligned} & \underline{\emptyset} \triangleright x =_{\text{Ax 2}} \\ & = \overline{x - x} \triangleright x =_{\text{Ax 3}} \\ & = x \triangleright x =_{\text{Ax 1}} \\ & = x \end{aligned} \quad (\text{A/8})$$

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## 2. Proofs based on Axioms 1–3 and 5

$$\begin{aligned}
 & \emptyset - x =_{A/8} \\
 & = \emptyset \triangleright \emptyset - x =_{Ax\ 3} \\
 & = \emptyset - x - \emptyset \triangleright \emptyset =_{Ax\ 2} \\
 & = \emptyset - x - \emptyset \triangleright \overline{\emptyset - \emptyset} =_{Ax\ 5} \\
 & = (\emptyset - x \triangleright \emptyset) - \emptyset =_{Ax\ 2} \\
 & = \overline{(\emptyset - x \triangleright x - x)} - \emptyset =_{Ax\ 5} \\
 & = \overline{(\emptyset \triangleright x) - x} - \emptyset =_{A/8} \\
 & = \overline{x - x} - \emptyset =_{Ax\ 2} \\
 & = \overline{\emptyset} - \emptyset =_{Ax\ 2} \\
 & = \emptyset
 \end{aligned} \tag{B/10}$$

$$\begin{aligned}
 & x \triangleright \emptyset =_{Ax\ 3} \\
 & = \overline{\emptyset - x} \triangleright x =_{B/10} \\
 & = \overline{\emptyset} \triangleright x =_{A/8} \\
 & = x
 \end{aligned} \tag{C/7}$$

$$\begin{aligned}
 & x - \emptyset =_{C/7} \\
 & = x - \emptyset \triangleright \emptyset =_{Ax\ 3} \\
 & = \emptyset \triangleright x =_{A/8} \\
 & = x
 \end{aligned} \tag{D/9}$$

$$\begin{aligned}
 & x @ x =_{@} \\
 & = x - \overline{(x - x)} =_{Ax\ 2} \\
 & = x - \overline{\emptyset} =_{D/9} \\
 & = x
 \end{aligned} \tag{E/13}$$

$$\begin{aligned}
 & (x \triangleright y) - x =_{Ax\ 5} \\
 & = \overline{x - x} \triangleright y - x =_{Ax\ 2} \\
 & = \overline{\emptyset} \triangleright y - x =_{A/8} \\
 & = y - x
 \end{aligned} \tag{F/}$$

$$\begin{aligned}
 & (x \triangleright y) - y =_{Ax\ 5} \\
 & = x - y \triangleright \overline{y - y} =_{Ax\ 2} \\
 & = x - y \triangleright \overline{\emptyset} =_{C/7} \\
 & = x - y
 \end{aligned} \tag{G/}$$

$$\begin{aligned}
& (x \triangleright y \triangleright z) - x =_{Ax 5} \\
& = \underline{(x \triangleright y) - x} \triangleright z - x =_{F/} \\
& = y - x \triangleright z - x
\end{aligned} \tag{H/18}$$

$$\begin{aligned}
& x \triangleright y - x =_{Ax 3} \\
& = \underline{y - x - x} \triangleright x =_{G/} \\
& = \underline{(y - x \triangleright x) - x} \triangleright x =_{Ax 3} \\
& = x \triangleright \underline{(y - x \triangleright x)} =_{Ax 3} \\
& = x \triangleright \underline{(x \triangleright y)} =_{Ax 3} \\
& = \underline{(x \triangleright y) - x} \triangleright x =_{F/} \\
& = y - x \triangleright x =_{Ax 3} \\
& = x \triangleright y
\end{aligned} \tag{I/31}$$

### 3. Proofs based on Axioms 1–5

$$\begin{aligned}
& x - y \triangleright x =_{I/31} \\
& = x - y \triangleright x - (x - y) =_{@, Ax 4} \\
& = x - \underline{(y - y)} =_{Ax 2} \\
& = x - \underline{0} =_{D/9} \\
& = x
\end{aligned} \tag{J/30}$$

$$\begin{aligned}
& \underline{x @ y} \triangleright x - y =_{@} \\
& = \underline{x - (x - y)} \triangleright x - y =_{Ax 3} \\
& = x - y \triangleright x =_{J/30} \\
& = x
\end{aligned} \tag{K/24}$$

$$\begin{aligned}
& x - (y - x) =_{Ax 4} \\
& = x - y \triangleright \underline{x @ x} =_{E/13} \\
& = x - y \triangleright \underline{x} =_{J/30} \\
& = x
\end{aligned} \tag{L/27}$$

$$\begin{aligned}
& x - y - z =_{C/7} \\
& = x - y - z \triangleright \underline{0} =_{Ax 2} \\
& = x - y - z \triangleright \underline{x - y - (x - y)} =_{L/27} \\
& = x - y - z \triangleright x - y - \underline{(x - y - (y - (x - y)))} =_{L/27} \\
& = x - y - z \triangleright x - y - (x - y - \underline{y}) =_{@, Ax 4} \\
& = x - y - (z - y)
\end{aligned} \tag{M/}$$

$$\begin{aligned}
& x - y - (z - x) =_{Ax\ 4,@} \\
& = x - y - z \triangleright x - y - \underline{(x - y - x)} =_{G/} \\
& = x - y - z \triangleright x - y - \underline{((x - y \triangleright x) - x)} =_{J/30} \\
& = x - y - z \triangleright x - y - \underline{(x - x)} =_{Ax\ 2} \\
& = x - y - z \triangleright \underline{x - y - \bar{0}} =_{D/9} \\
& = x - y - z \triangleright \underline{x - y} =_{J/30} \\
& = x - y
\end{aligned} \tag{N/}$$

$$\begin{aligned}
& x - (y - (z - x)) =_{Ax\ 4,@} \\
& = x - y \triangleright x - \underline{(x - (z - x))} =_{L/27} \\
& = x - y \triangleright \underline{x - x} =_{Ax\ 2} \\
& = x - y \triangleright \bar{0} =_{C/7} \\
& = x - y
\end{aligned} \tag{O/}$$

$$\begin{aligned}
& x - z - y =_{M/} \\
& = x - z - (y - z) =_{F/} \\
& = \underline{(y - z \triangleright x - z)} - \underline{(y - z)} =_{Ax\ 5,N/} \\
& = \underline{(y \triangleright x) - z} - \underline{(y - z - (x - y))} =_{Ax\ 3,M/} \\
& = \underline{(x - y \triangleright y) - z} - \underline{(y - z - (x - y - z))} =_{Ax\ 5} \\
& = \underline{(x - y - z \triangleright y - z)} - \underline{(y - z - (x - y - z))} =_{Ax\ 3} \\
& = \underline{(y - z - (x - y - z) \triangleright x - y - z)} - \underline{(y - z - (x - y - z))} =_{F/} \\
& = x - y - z - (y - z - (x - y - z)) =_{L/27} \\
& = x - y - z
\end{aligned} \tag{P/19}$$

$$\begin{aligned}
& x - (y \triangleright z) =_{L/27,@} \\
& = x - (y \triangleright z) - \underline{(y - (x - (y \triangleright z)))} =_{Ax\ 4} \\
& = x - (y \triangleright z) - \underline{(y - x \triangleright y - (y - (y \triangleright z)))} =_{G/} \\
& = x - (y \triangleright z) - (y - x \triangleright y - \underline{((y \triangleright (y \triangleright z)) - (y \triangleright z))}) =_{Ax\ 3} \\
& = x - (y \triangleright z) - (y - x \triangleright y - \underline{((y \triangleright z) - y \triangleright y) - (y \triangleright z)}) =_{F/} \\
& = x - (y \triangleright z) - (y - x \triangleright y - \underline{((z - y \triangleright y) - (y \triangleright z))}) =_{Ax\ 3} \\
& = x - (y \triangleright z) - (y - x \triangleright y - \underline{((y \triangleright z) - (y \triangleright z))}) =_{Ax\ 2} \\
& = x - (y \triangleright z) - (y - x \triangleright \underline{y - \bar{0}}) =_{D/9} \\
& = x - (y \triangleright z) - \underline{(y - x \triangleright y)} =_{J/30} \\
& = x - (y \triangleright z) - \underline{y} =_{P/19} \\
& = x - y - \underline{(y \triangleright z)} =_{L/27} \\
& = x - y - \underline{(y - (x - y) \triangleright z)} =_{O/} \\
& = x - y - \underline{((y - (x - y) \triangleright z) - (y - (x - y)))} =_{F/} \\
& = x - y - \underline{(z - (y - (x - y)))} =_{O/} \\
& = x - y - z
\end{aligned} \tag{Q/26}$$

$$\begin{aligned}
& \underline{x - x @ y} =_{K/24, @} \\
= & \underline{(x - (x - y) \triangleright x - y) - (x - (x - y))} =_{F/} \\
& = x - y - (x - (x - y)) =_{L/27} \\
& = x - y
\end{aligned} \tag{R/}$$

$$\begin{aligned}
& x \triangleright y - z =_{I/31} \\
= & x \triangleright \underline{y - z - x} =_{P/19} \\
= & x \triangleright \underline{y - x - z} =_{Q/26} \\
= & x \triangleright \underline{y - (x \triangleright z)} =_{I/31} \\
= & x \triangleright \underline{y - (x \triangleright z - x)} =_{Q/26} \\
= & x \triangleright \underline{y - x - (z - x)} =_{P/19} \\
= & x \triangleright \underline{y - (z - x) - x} =_{I/31} \\
= & \underline{x \triangleright y - (z - x)} =_{L/27} \\
= & \underline{x - (z - x) \triangleright y - (z - x)} =_{Ax 5} \\
= & (x \triangleright y) - (z - x)
\end{aligned} \tag{S/}$$

$$\begin{aligned}
& x - y \triangleright x - z =_{S/} \\
= & \underline{(x - y \triangleright x) - (z - (x - y))} =_{J/30} \\
= & \underline{x - (z - (x - y))} =_{Ax 4, @} \\
= & x - z \triangleright \underline{x - (x - (x - y))} =_{@, R/} \\
= & x - z \triangleright x - y
\end{aligned} \tag{T/21}$$

$$\begin{aligned}
& x - y \triangleright \underline{x @ y} =_{@} \\
= & x - y \triangleright \underline{x - (x - y)} =_{I/31} \\
= & x - y \triangleright x =_{J/30} \\
= & x
\end{aligned} \tag{U/25}$$

$$\begin{aligned}
& x \triangleright \underline{y @ x} =_{@} \\
= & x \triangleright \underline{y - (y - x)} =_{Ax 3} \\
= & \underline{y - (y - x) - x} \triangleright x =_{P/19} \\
= & \underline{y - x - (y - x)} \triangleright x =_{Ax 2} \\
= & \underline{\emptyset} \triangleright x =_{A/8} \\
= & x
\end{aligned} \tag{V/28}$$

$$\begin{aligned}
& x \triangleright x - y =_{Ax\ 3} \\
& = \underline{x - y - x} \triangleright x =_{G/} \\
& = \underline{(x - y \triangleright x) - x} \triangleright x =_{J/30} \\
& = \underline{x - x} \triangleright x =_{Ax\ 2} \\
& = \underline{\emptyset} \triangleright x =_{A/8} \\
& = x
\end{aligned}$$

(W/29)

$$\begin{aligned}
& \underline{x @ y} - y =_{@} \\
& = \underline{x - (x - y)} - y =_{P/19} \\
& = x - y - (x - y) =_{Ax\ 2} \\
& = \emptyset
\end{aligned}$$

(X/6)

$$\begin{aligned}
& x \triangleright y \triangleright z =_{U/25} \\
& = \underline{(x \triangleright y \triangleright z) - x} \triangleright (x \triangleright y \triangleright z) @ x =_{H/18} \\
& = \underline{y - x \triangleright z - x} \triangleright (x \triangleright y \triangleright z) @ x =_{Ax\ 5, @} \\
& = \underline{(y \triangleright z) - x} \triangleright \underline{(x \triangleright y \triangleright z) - ((x \triangleright y \triangleright z) - x)} =_{H/18} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright y \triangleright z) - (y - x) - (z - x)} =_{Q/26} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright y \triangleright z) - (y - x) - (z - x)} =_{Ax\ 5} \\
& = (y \triangleright z) - x \triangleright \underline{((x \triangleright y) - (y - x) \triangleright z - (y - x)) - (z - x)} =_{Ax\ 5} \\
& = (y \triangleright z) - x \triangleright \underline{(x - (y - x) \triangleright y - (y - x) \triangleright z - (y - x)) - (z - x)} =_{L/27} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright y - (y - x) \triangleright z - (y - x)) - (z - x)} =_{V/28} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright z - (y - x)) - (z - x)} =_{L/27} \\
& = (y \triangleright z) - x \triangleright \underline{(x - (y - x) \triangleright z - (y - x)) - (z - x)} =_{Ax\ 5} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright z) - (y - x) - (z - x)} =_{P/19} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright z) - (z - x) - (y - x)} =_{Ax\ 5} \\
& = (y \triangleright z) - x \triangleright \underline{(x - (z - x) \triangleright z - (z - x)) - (y - x)} =_{L/27} \\
& = (y \triangleright z) - x \triangleright \underline{(x \triangleright z - (z - x)) - (y - x)} =_{V/28} \\
& = (y \triangleright z) - x \triangleright \underline{x - (y - x)} =_{L/27} \\
& = (y \triangleright z) - x \triangleright x =_{Ax\ 3} \\
& = x \triangleright (y \triangleright z)
\end{aligned}$$

(Y/11)

$$\begin{aligned}
& x - (y - z) =_{Ax\ 4, @} \\
& = x - y \triangleright x - (x - z) =_{T/21} \\
& = \underline{x - (x - z)} \triangleright x - y =_{@} \\
& = \underline{x @ z} \triangleright x - y
\end{aligned}$$

(Z/)

$$\begin{aligned}
& x @ (y @ z) = @ \\
& = x @ (y - (y - z)) = @ \\
& = x - \underline{(x - (y - (y - z)))} = Z/ \\
& = x - \underline{(x @ (y - z) \triangleright x - y)} = Q/26 \\
& = x - x @ (y - z) - (x - y) = P/19 \\
& = x - (x - y) - \underline{x @ (y - z)} = @ \\
& = x - (x - y) - \underline{(x - (x - (y - z)))} = Z/ \\
& = x - (x - y) - \underline{(x - (x @ z \triangleright x - y))} = Q/26 \\
& = x - (x - y) - \underline{(x - x @ z - (x - y))} = Q/26 \\
& = x - (x - y \triangleright \underline{x - x @ z} - (x - y)) = R/ \\
& = x - (x - y \triangleright \underline{x - z - (x - y)}) = P/19 \\
& = x - (x - y \triangleright \underline{x - (x - y) - z}) = Q/26 \\
& = x - (x - y) - (x - (x - y) - z) = @ \\
& = \underline{(x - (x - y))} @ z = @ \\
& = \underline{x @ y} @ z
\end{aligned}$$

(a/12)

$$\begin{aligned}
& \underline{x - (y - z)} - (x - z) = Z/ \\
& = \underline{(x @ z \triangleright x - y)} - (x - z) = Ax 5 \\
& = x @ z - \underline{(x - z) \triangleright x - y - (x - z)} = L/27, P/19 \\
& = x @ z - \underline{(x - z - (x - (x - z))) \triangleright x - (x - z) - y} = @, @ \\
& = \underline{x @ z - (x - z - x @ z)} \triangleright x @ z - y = L/27 \\
& = x @ z \triangleright x @ z - y = W/29 \\
& = x @ z
\end{aligned}$$

(b/)

$$\begin{aligned}
& (x \triangleright y) @ z = @ \\
& = (x \triangleright y) - \underline{((x \triangleright y) - z)} = Ax 5 \\
& = (x \triangleright y) - \underline{(x - z \triangleright y - z)} = Ax 5 \\
& = \underline{x - (x - z \triangleright y - z)} \triangleright \underline{y - (x - z \triangleright y - z)} = Q/26, Q/26 \\
& = \underline{x - (x - z) - (y - z)} \triangleright \underline{y - (x - z) - (y - z)} = P/19 \\
& = \underline{x - (y - z) - (x - z)} \triangleright \underline{y - (x - z) - (y - z)} = b/, b/ \\
& = \underline{x @ z} \triangleright \underline{y @ z}
\end{aligned}$$

(c/14)

$$\begin{aligned}
& x @ (y \triangleright z) = @ \\
& = x - \underline{(x - (y \triangleright z))} = Q/26 \\
& = x - \underline{(x - y - z)} = Z/, @ \\
& = x - (x - z) \triangleright x - (x - y) = T/21 \\
& = \underline{x - (x - y)} \triangleright \underline{x - (x - z)} = @ \\
& = \underline{x @ y} \triangleright \underline{x @ z}
\end{aligned}$$

(d/15)

$$\begin{aligned}
& \underline{x @ y - z} = @ \\
& = x - (x - y) - z = P/19 \\
& = x - z - (x - y) = Q/26 \\
& = x - \underline{(z \triangleright x - y)} = I/31 \\
& = x - \underline{(z \triangleright x - y - z)} = P/19 \\
& = x - (z \triangleright \underline{x - z - y}) = Q/26 \\
& = x - (z \triangleright \underline{x - (z \triangleright y)}) = I/31 \\
& = x - (z \triangleright x - \underline{(z \triangleright y - z)}) = Q/26 \\
& = x - z - \underline{(x - (z \triangleright y - z))} = Q/26 \\
& = x - z - \underline{(x - z - (y - z))} = @ \\
& = (x - z) @ (y - z)
\end{aligned} \tag{e/16}$$

$$\begin{aligned}
& \underline{x \triangleright y @ z} = V/28 \\
& = \underline{x \triangleright y @ x \triangleright y @ z} = Y/11 \\
& = \underline{x \triangleright (y @ x \triangleright y @ z)} = W/29, d/15 \\
& = \underline{x \triangleright x - (x - z)} \triangleright y @ (x \triangleright z) = E/13, @ \\
& = \underline{x @ x \triangleright x @ z} \triangleright y @ (x \triangleright z) = d/15 \\
& = \underline{x @ (x \triangleright z)} \triangleright y @ (x \triangleright z) = c/14 \\
& = (x \triangleright y) @ (x \triangleright z)
\end{aligned} \tag{f/17}$$

$$\begin{aligned}
& x - (y \triangleright z) = Q/26 \\
& = x - y - z = P/19 \\
& = x - z - y = Q/26 \\
& = x - (z \triangleright y)
\end{aligned} \tag{g/20}$$

$$\begin{aligned}
& \underline{x @ y @ z} = @ \\
& = \underline{(x - (x - y))} @ z = @ \\
& = x - (x - y) - (x - (x - y) - z) = Q/26 \\
& = x - (x - y \triangleright \underline{x - (x - y) - z}) = P/19 \\
& = x - \underline{(x - y \triangleright x - z - (x - y))} = I/31 \\
& = x - \underline{(x - y \triangleright x - z)} = Q/26 \\
& = x - (x - y) - (x - z)
\end{aligned} \tag{h/}$$

$$\begin{aligned}
& x @ y @ z = h/ \\
& = x - (x - y) - (x - z) = P/19 \\
& = x - (x - z) - (x - y) = h/ \\
& = x @ z @ y
\end{aligned} \tag{i/22}$$

$$\begin{aligned}
& x - y \triangleright x - z =_{T/21} \\
& = x - z \triangleright x - y =_{S/} \\
& = \frac{(x - z \triangleright x) - (y - (x - z))}{=} =_{J/30,Z/} \\
& = \frac{x - (y @ z \triangleright y - x)}{=} =_{S/} \\
& = x - \frac{(y @ z \triangleright y) - (x - y @ z)}{=} =_{@} \\
& = \frac{x - ((y - (y - z)) \triangleright y) - (x - y @ z)}{=} =_{J/30,J/30} \\
& = \frac{(x - y @ z \triangleright x) - (y - (x - y @ z))}{=} =_{S/} \\
& = x - y @ z \triangleright x - y =_{T/21} \\
& = x - y \triangleright x - y @ z =_{S/} \\
& = \frac{(x - y \triangleright x) - (y @ z - (x - y))}{=} =_{J/30,@} \\
& = \frac{x - (y - (y - z)) - (x - y)}{=} =_{N/} \\
& = \frac{x - (y - (y - z))}{=} =_{@} \\
& = x - y @ z
\end{aligned} \tag{j/}$$

$$\begin{aligned}
& x - y @ z =_{j/} \\
& = x - y \triangleright x - z =_{T/21} \\
& = x - z \triangleright x - y =_{j/} \\
& = x - z @ y
\end{aligned} \tag{k/23}$$

$$\begin{aligned}
& (x - y) @ z =_{@} \\
& = x - y - \frac{(x - y - z)}{=} =_{P/19} \\
& = x - y - \frac{(x - z - y)}{=} =_{Q/26} \\
& = x - \frac{(y \triangleright x - z - y)}{=} =_{I/31} \\
& = x - \frac{(y \triangleright x - z)}{=} =_{Q/26} \\
& = x - y - (x - z) =_{P/19} \\
& = \frac{x - (x - z) - y}{=} =_{@} \\
& = x @ z - y
\end{aligned} \tag{l/32}$$

$$\begin{aligned}
& x @ (y - z) =_{@} \\
& = \frac{x - (x - (y - z))}{=} =_{Z/} \\
& = x - \frac{(x - (x - z)) \triangleright x - y}{=} =_{Q/26} \\
& = \frac{x - (x - (x - z)) - (x - y)}{=} =_{R/} \\
& = \frac{x - z - (x - y)}{=} =_{P/19} \\
& = \frac{x - (x - y) - z}{=} =_{@} \\
& = x @ y - z
\end{aligned} \tag{m/33}$$