The Axiomatization of Overriding and Update[☆]
Online Appendix: Human-Readable Proofs

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Abstract

This document contains human-readable proofs for the derived laws in Figure 3 of the article published under the same title in the Journal of Applied Logic. The proofs are mostly based on the ones provided by ISABELLE.

In the following, we provide equational proofs for all laws in Figure 3. Lemmas are tagged with the number of the law and with a letter to indicate the order of proofs: in each proof, only laws with an earlier letter may be used. If a rule is only applied to a part of the formula, the part that is going to be replaced is red and underlined, and the part that has just been replaced is blue and overlined. The proofs are mostly based on the ones provided by ISABELLE; they differ mainly if the automatic proof for one law contains (most of) the proof for another.

We assume left-associativity and the usual priorities: @ binds strongest and ▷ binds weakest.

The axioms are:

$$x \triangleright x = x \qquad \text{(Ax 1)}$$
$$x - x = \emptyset \qquad \text{(Ax 2)}$$
$$x - y \triangleright y = y \triangleright x \qquad \text{(Ax 3)}$$
$$x - (y - z) = x - y \triangleright x \,@\, z \qquad \text{(Ax 4)}$$
$$(x \triangleright y) - z = x - z \triangleright y - z \qquad \text{(Ax 5)}$$

The operator @ is defined as an abbreviation:

$$x \,@\, y = x - (x - y) \qquad \text{(@)}$$

1. Proof based on Axioms 1–3

$$
\begin{aligned}
\underline{\emptyset} \triangleright x &=_{\text{Ax 2}} \\
= \overline{x - x} \triangleright x &=_{\text{Ax 3}} \\
= x \triangleright x &=_{\text{Ax 1}} \\
= x &\qquad \text{(A/8)}
\end{aligned}
$$

2. Proofs based on Axioms 1–3 and 5

$$\emptyset - x =_{A/8}$$
$$= \emptyset \triangleright \emptyset - x =_{Ax\,3}$$
$$= \emptyset - x - \emptyset \triangleright \underline{\emptyset} =_{Ax\,2}$$
$$= \emptyset - x - \emptyset \triangleright \overline{\emptyset - \emptyset} =_{Ax\,5}$$
$$= (\emptyset - x \triangleright \underline{\emptyset}) - \emptyset =_{Ax\,2}$$
$$= (\emptyset - x \triangleright \overline{x - x}) - \emptyset =_{Ax\,5}$$
$$= \overline{(\emptyset \triangleright x) - x} - \emptyset =_{A/8}$$
$$= \overline{x - x} - \emptyset =_{Ax\,2}$$
$$= \overline{\emptyset} - \emptyset =_{Ax\,2}$$
$$= \emptyset \qquad\qquad\qquad (B/10)$$

$$x \triangleright \emptyset =_{Ax\,3}$$
$$= \underline{\emptyset - x} \triangleright x =_{B/10}$$
$$= \overline{\emptyset} \triangleright x =_{A/8}$$
$$= x \qquad\qquad\qquad (C/7)$$

$$x - \emptyset =_{C/7}$$
$$= x - \emptyset \triangleright \emptyset =_{Ax\,3}$$
$$= \emptyset \triangleright x =_{A/8}$$
$$= x \qquad\qquad\qquad (D/9)$$

$$x \,@\, x =_{@}$$
$$= x - \underline{(x - x)} =_{Ax\,2}$$
$$= x - \overline{\emptyset} =_{D/9}$$
$$= x \qquad\qquad\qquad (E/13)$$

$$(x \triangleright y) - x =_{Ax\,5}$$
$$= \underline{x - x} \triangleright y - x =_{Ax\,2}$$
$$= \overline{\emptyset} \triangleright y - x =_{A/8}$$
$$= y - x \qquad\qquad\qquad (F/)$$

$$(x \triangleright y) - y =_{Ax\,5}$$
$$= x - y \triangleright \underline{y - y} =_{Ax\,2}$$
$$= x - y \triangleright \overline{\emptyset} =_{C/7}$$
$$= x - y \qquad\qquad\qquad (G/)$$

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$$(x \triangleright y \triangleright z) - x =_{\text{Ax 5}}$$
$$= (x \triangleright y) - x \triangleright z - x =_{\text{F/}}$$
$$= y - x \triangleright z - x \tag{H/18}$$

$$x \triangleright y - x =_{\text{Ax 3}}$$
$$= y - x - x \triangleright x =_{\text{G/}}$$
$$= (y - x \triangleright x) - x \triangleright x =_{\text{Ax 3}}$$
$$= x \triangleright (y - x \triangleright x) =_{\text{Ax 3}}$$
$$= x \triangleright (x \triangleright y) =_{\text{Ax 3}}$$
$$= (x \triangleright y) - x \triangleright x =_{\text{F/}}$$
$$= y - x \triangleright x =_{\text{Ax 3}}$$
$$= x \triangleright y \tag{I/31}$$

3. Proofs based on Axioms 1–5

$$x - y \triangleright x =_{\text{I/31}}$$
$$= x - y \triangleright x - (x - y) =_{@,\text{Ax 4}}$$
$$= x - (y - y) =_{\text{Ax 2}}$$
$$= x - \overline{\emptyset} =_{\text{D/9}}$$
$$= x \tag{J/30}$$

$$x \,@\, y \triangleright x - y =_{@}$$
$$= x - (x - y) \triangleright x - y =_{\text{Ax 3}}$$
$$= x - y \triangleright x =_{\text{J/30}}$$
$$= x \tag{K/24}$$

$$x - (y - x) =_{\text{Ax 4}}$$
$$= x - y \triangleright x \,@\, x =_{\text{E/13}}$$
$$= x - y \triangleright x =_{\text{J/30}}$$
$$= x \tag{L/27}$$

$$x - y - z =_{\text{C/7}}$$
$$= x - y - z \triangleright \emptyset =_{\text{Ax 2}}$$
$$= x - y - z \triangleright x - y - (x - y) =_{\text{L/27}}$$
$$= x - y - z \triangleright x - y - (x - y - (y - (x - y))) =_{\text{L/27}}$$
$$= x - y - z \triangleright x - y - (x - y - y) =_{@,\text{Ax 4}}$$
$$= x - y - (z - y) \tag{M/}$$

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$$x - y - (z - x) =_{\text{Ax 4},@}$$
$$= x - y - z \triangleright x - y - (x - y - x) =_{\text{G/}}$$
$$= x - y - z \triangleright x - y - \overline{((x - y \triangleright x) - x)} =_{\text{J/30}}$$
$$= x - y - z \triangleright x - y - \overline{(x - x)} =_{\text{Ax 2}}$$
$$= x - y - z \triangleright x - y - \overline{\overline{\emptyset}} =_{\text{D/9}}$$
$$= x - y - z \triangleright x - y =_{\text{J/30}}$$
$$= x - y \qquad\qquad (\text{N/})$$

$$x - (y - (z - x)) =_{\text{Ax 4},@}$$
$$= x - y \triangleright x - (x - (z - x)) =_{\text{L/27}}$$
$$= x - y \triangleright x - x =_{\text{Ax 2}}$$
$$= x - y \triangleright \overline{\emptyset} =_{\text{C/7}}$$
$$= x - y \qquad\qquad (\text{O/})$$

$$x - z - y =_{\text{M/}}$$
$$= x - z - (y - z) =_{\text{F/}}$$
$$= (y - z \triangleright x - z) - (y - z) =_{\text{Ax 5},\text{N/}}$$
$$= (y \triangleright x) - z - (y - z - (x - y)) =_{\text{Ax 3},\text{M/}}$$
$$= (x - y \triangleright y) - z - (y - z - (x - y - z)) =_{\text{Ax 5}}$$
$$= (x - y - z \triangleright y - z) - (y - z - (x - y - z)) =_{\text{Ax 3}}$$
$$= (y - z - (x - y - z) \triangleright x - y - z) - (y - z - (x - y - z)) =_{\text{F/}}$$
$$= x - y - z - (y - z - (x - y - z)) =_{\text{L/27}}$$
$$= x - y - z \qquad\qquad (\text{P/19})$$

$$x - (y \triangleright z) =_{\text{L/27},@}$$
$$= x - (y \triangleright z) - (y - (x - (y \triangleright z))) =_{\text{Ax 4}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - (y - (y \triangleright z))) =_{\text{G/}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - ((y \triangleright (y \triangleright z)) - (y \triangleright z))) =_{\text{Ax 3}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - (((y \triangleright z) - y \triangleright y) - (y \triangleright z))) =_{\text{F/}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - ((z - y \triangleright y) - (y \triangleright z))) =_{\text{Ax 3}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - ((y \triangleright z) - (y \triangleright z))) =_{\text{Ax 2}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y - \overline{\emptyset}) =_{\text{D/9}}$$
$$= x - (y \triangleright z) - (y - x \triangleright y) =_{\text{J/30}}$$
$$= x - (y \triangleright z) - y =_{\text{P/19}}$$
$$= x - y - (y \triangleright z) =_{\text{L/27}}$$
$$= x - y - (y - (x - y) \triangleright z) =_{\text{O/}}$$
$$= x - y - ((y - (x - y) \triangleright z) - (y - (x - y))) =_{\text{F/}}$$
$$= x - y - (z - (y - (x - y))) =_{\text{O/}}$$
$$= x - y - z \qquad\qquad (\text{Q/26})$$

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$$\underline{x} - \underline{x \mathbin{@} y} =_{\text{K}/24,\,@}$$
$$= \overline{(x - (x - y) \triangleright x - y) - (x - (x - y))} =_{\text{F}/}$$
$$= x - y - (x - (x - y)) =_{\text{L}/27}$$
$$= x - y \tag{R/}$$

$$x \triangleright y - z =_{\text{I}/31}$$
$$= x \triangleright \underline{y - z - x} =_{\text{P}/19}$$
$$= x \triangleright \underline{y - x - z} =_{\text{Q}/26}$$
$$= x \triangleright y - \underline{(x \triangleright z)} =_{\text{I}/31}$$
$$= x \triangleright y - \underline{(x \triangleright z - x)} =_{\text{Q}/26}$$
$$= x \triangleright \underline{y - x - (z - x)} =_{\text{P}/19}$$
$$= x \triangleright y - (z - x) - x =_{\text{I}/31}$$
$$= \underline{x} \triangleright y - (z - x) =_{\text{L}/27}$$
$$= \overline{x - (z - x)} \triangleright y - (z - x) =_{\text{Ax }5}$$
$$= (x \triangleright y) - (z - x) \tag{S/}$$

$$x - y \triangleright x - z =_{\text{S}/}$$
$$= \underline{(x - y \triangleright x)} - (z - (x - y)) =_{\text{J}/30}$$
$$= \underline{x} - (z - (x - y)) =_{\text{Ax }4,\,@}$$
$$= x - z \triangleright \underline{x - (x - (x - y))} =_{@,\text{R}/}$$
$$= x - z \triangleright \overline{x - y} \tag{T/21}$$

$$x - y \triangleright \underline{x \mathbin{@} y} =_{@}$$
$$= x - y \triangleright \overline{x - (x - y)} =_{\text{I}/31}$$
$$= x - y \triangleright x =_{\text{J}/30}$$
$$= x \tag{U/25}$$

$$x \triangleright \underline{y \mathbin{@} x} =_{@}$$
$$= x \triangleright \overline{y - (y - x)} =_{\text{Ax }3}$$
$$= \underline{y - (y - x) - x} \triangleright x =_{\text{P}/19}$$
$$= \overline{y - x - (y - x)} \triangleright x =_{\text{Ax }2}$$
$$= \overline{\emptyset} \triangleright x =_{\text{A}/8}$$
$$= x \tag{V/28}$$

$$x \triangleright x - y =_{\text{Ax 3}}$$
$$= x - y - x \triangleright x =_{\text{G/}}$$
$$= (x - y \triangleright x) - x \triangleright x =_{\text{J/30}}$$
$$= x - x \triangleright x =_{\text{Ax 2}}$$
$$= \overline{\emptyset} \triangleright x =_{\text{A/8}}$$
$$= x \qquad \qquad \text{(W/29)}$$

$$x \,@\, y - y =_{@}$$
$$= x - (x - y) - y =_{\text{P/19}}$$
$$= x - y - (x - y) =_{\text{Ax 2}}$$
$$= \emptyset \qquad \qquad \text{(X/6)}$$

$$x \triangleright y \triangleright z =_{\text{U/25}}$$
$$= (x \triangleright y \triangleright z) - x \triangleright (x \triangleright y \triangleright z) \,@\, x =_{\text{H/18}}$$
$$= y - x \triangleright z - x \triangleright (x \triangleright y \triangleright z) \,@\, x =_{\text{Ax 5},@}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright y \triangleright z) - ((x \triangleright y \triangleright z) - x) =_{\text{H/18}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright y \triangleright z) - (y - x \triangleright z - x) =_{\text{Q/26}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright y \triangleright z) - (y - x) - (z - x) =_{\text{Ax 5}}$$
$$= (y \triangleright z) - x \triangleright ((x \triangleright y) - (y - x) \triangleright z - (y - x)) - (z - x) =_{\text{Ax 5}}$$
$$= (y \triangleright z) - x \triangleright (x - (y - x) \triangleright y - (y - x) \triangleright z - (y - x)) - (z - x) =_{\text{L/27}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright y - (y - x) \triangleright z - (y - x)) - (z - x) =_{\text{V/28}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright z - (y - x)) - (z - x) =_{\text{L/27}}$$
$$= (y \triangleright z) - x \triangleright (x - (y - x) \triangleright z - (y - x)) - (z - x) =_{\text{Ax 5}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright z) - (y - x) - (z - x) =_{\text{P/19}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright z) - (z - x) - (y - x) =_{\text{Ax 5}}$$
$$= (y \triangleright z) - x \triangleright (x - (z - x) \triangleright z - (z - x)) - (y - x) =_{\text{L/27}}$$
$$= (y \triangleright z) - x \triangleright (x \triangleright z - (z - x)) - (y - x) =_{\text{V/28}}$$
$$= (y \triangleright z) - x \triangleright x - (y - x) =_{\text{L/27}}$$
$$= (y \triangleright z) - x \triangleright x =_{\text{Ax 3}}$$
$$= x \triangleright (y \triangleright z) \qquad \qquad \text{(Y/11)}$$

$$x - (y - z) =_{\text{Ax 4},@}$$
$$= x - y \triangleright x - (x - z) =_{\text{T/21}}$$
$$= x - (x - z) \triangleright x - y =_{@}$$
$$= x \,@\, z \triangleright x - y \qquad \qquad \text{(Z/)}$$

$$x \mathbin{@} (y \mathbin{@} z) =_{@}$$
$$= x \mathbin{@} \overline{(y - (y - z))} =_{@}$$
$$= x - \underline{(x - (y - (y - z)))} =_{Z/}$$
$$= x - \overline{(x \mathbin{@} (y - z) \mathbin{\triangleright} x - y)} =_{Q/26}$$
$$= x - x \mathbin{@} (y - z) - (x - y) =_{P/19}$$
$$= x - (x - y) - \underline{x \mathbin{@} (y - z)} =_{@}$$
$$= x - (x - y) - \overline{(x - (x - (y - z)))} =_{Z/}$$
$$= x - (x - y) - \underline{(x - (x \mathbin{@} z \mathbin{\triangleright} x - y))} =_{Q/26}$$
$$= x - (x - y) - \overline{(x - x \mathbin{@} z - (x - y))} =_{Q/26}$$
$$= x - (x - y \mathbin{\triangleright} \underline{x - x \mathbin{@} z} - (x - y)) =_{R/}$$
$$= x - (x - y \mathbin{\triangleright} \underline{x - z - (x - y)}) =_{P/19}$$
$$= x - (x - y \mathbin{\triangleright} \overline{x - (x - y) - z}) =_{Q/26}$$
$$= x - (x - y) - (x - (x - y) - z) =_{@}$$
$$= \underline{(x - (x - y))} \mathbin{@} z =_{@}$$
$$= \overline{x \mathbin{@} y} \mathbin{@} z \tag{a/12}$$

$$\underline{x - (y - z)} - (x - z) =_{Z/}$$
$$= \overline{(x \mathbin{@} z \mathbin{\triangleright} x - y)} - (x - z) =_{Ax\,5}$$
$$= x \mathbin{@} z - \underline{(x - z)} \mathbin{\triangleright} \underline{x - y - (x - z)} =_{L/27,P/19}$$
$$= x \mathbin{@} z - \overline{(x - z - (x - (x - z)))} \mathbin{\triangleright} \overline{x - (x - z) - y} =_{@,@}$$
$$= \underline{x \mathbin{@} z - (x - z - \overline{x \mathbin{@} z})} \mathbin{\triangleright} \overline{x \mathbin{@} z} - y =_{L/27}$$
$$= \overline{x \mathbin{@} z} \mathbin{\triangleright} x \mathbin{@} z - y =_{W/29}$$
$$= x \mathbin{@} z \tag{b/}$$

$$(x \mathbin{\triangleright} y) \mathbin{@} z =_{@}$$
$$= (x \mathbin{\triangleright} y) - \underline{((x \mathbin{\triangleright} y) - z)} =_{Ax\,5}$$
$$= (x \mathbin{\triangleright} y) - \overline{(x - z \mathbin{\triangleright} y - z)} =_{Ax\,5}$$
$$= \underline{x - (x - z \mathbin{\triangleright} y - z)} \mathbin{\triangleright} \underline{y - (x - z \mathbin{\triangleright} y - z)} =_{Q/26,Q/26}$$
$$= \overline{x - (x - z) - (y - z)} \mathbin{\triangleright} \overline{y - (x - z) - (y - z)} =_{P/19}$$
$$= \underline{x - (y - z) - (x - z)} \mathbin{\triangleright} \overline{y - (x - z) - (y - z)} =_{b/,b/}$$
$$= \overline{x \mathbin{@} z} \mathbin{\triangleright} \overline{y \mathbin{@} z} \tag{c/14}$$

$$x \mathbin{@} (y \mathbin{\triangleright} z) =_{@}$$
$$= x - \underline{(x - (y \mathbin{\triangleright} z))} =_{Q/26}$$
$$= x - \overline{(x - y - z)} =_{Z/,@}$$
$$= x - (x - z) \mathbin{\triangleright} x - (x - y) =_{T/21}$$
$$= \underline{x - (x - y)} \mathbin{\triangleright} \underline{x - (x - z)} =_{@}$$
$$= \overline{x \mathbin{@} y} \mathbin{\triangleright} \overline{x \mathbin{@} z} \tag{d/15}$$

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$$x @ y - z =_@$$
$$= x - (x - y) - z =_{P/19}$$
$$= x - z - (x - y) =_{Q/26}$$
$$= x - (z \triangleright x - y) =_{I/31}$$
$$= x - (z \triangleright x - y - z) =_{P/19}$$
$$= x - (z \triangleright x - z - y) =_{Q/26}$$
$$= x - (z \triangleright x - (z \triangleright y)) =_{I/31}$$
$$= x - (z \triangleright x - (z \triangleright y - z)) =_{Q/26}$$
$$= x - z - (x - (z \triangleright y - z)) =_{Q/26}$$
$$= x - z - (x - z - (y - z)) =_@$$
$$= (x - z) @ (y - z) \tag{e/16}$$

$$x \triangleright y @ z =_{V/28}$$
$$= x \triangleright y @ x \triangleright y @ z =_{Y/11}$$
$$= x \triangleright (y @ x \triangleright y @ z) =_{W/29, d/15}$$
$$= x \triangleright x - (x - z) \triangleright y @ (x \triangleright z) =_{E/13, @}$$
$$= x @ x \triangleright x @ z \triangleright y @ (x \triangleright z) =_{d/15}$$
$$= x @ (x \triangleright z) \triangleright y @ (x \triangleright z) =_{c/14}$$
$$= (x \triangleright y) @ (x \triangleright z) \tag{f/17}$$

$$x - (y \triangleright z) =_{Q/26}$$
$$= x - y - z =_{P/19}$$
$$= x - z - y =_{Q/26}$$
$$= x - (z \triangleright y) \tag{g/20}$$

$$x @ y @ z =_@$$
$$= (x - (x - y)) @ z =_@$$
$$= x - (x - y) - (x - (x - y) - z) =_{Q/26}$$
$$= x - (x - y \triangleright x - (x - y) - z) =_{P/19}$$
$$= x - (x - y \triangleright x - z - (x - y)) =_{I/31}$$
$$= x - (x - y \triangleright x - z) =_{Q/26}$$
$$= x - (x - y) - (x - z) \tag{h/}$$

$$x @ y @ z =_{h/}$$
$$= x - (x - y) - (x - z) =_{P/19}$$
$$= x - (x - z) - (x - y) =_{h/}$$
$$= x @ z @ y \tag{i/22}$$

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$$x - y \triangleright x - z =_{T/21}$$
$$= x - z \triangleright x - y =_{S/}$$
$$= \underline{(x - z \triangleright x)} - \underline{(y - (x - z))} =_{J/30,Z/}$$
$$= \underline{x} - \overline{\underline{(y @ z \triangleright y - x)}} =_{S/}$$
$$= x - \overline{\underline{((y @ z \triangleright y) - (x - y @ z))}} =_{@}$$
$$= \underline{x} - (\overline{\underline{(y - (y - z) \triangleright y)}} - (x - y @ z)) =_{J/30,J/30}$$
$$= \overline{\underline{(x - y @ z \triangleright x)}} - (\overline{y} - (x - y @ z)) =_{S/}$$
$$= x - y @ z \triangleright x - y =_{T/21}$$
$$= x - y \triangleright x - y @ z =_{S/}$$
$$= \underline{(x - y \triangleright x)} - \underline{(y @ z - (x - y))} =_{J/30,@}$$
$$= \underline{x} - \overline{\underline{(y - (y - z) - (x - y))}} =_{N/}$$
$$= x - \overline{\underline{(y - (y - z))}} =_{@}$$
$$= x - \overline{y @ z} \tag{j/}$$

$$x - y @ z =_{j/}$$
$$= x - y \triangleright x - z =_{T/21}$$
$$= x - z \triangleright x - y =_{j/}$$
$$= x - z @ y \tag{k/23}$$

$$(x - y) @ z =_{@}$$
$$= x - y - \underline{(x - y - z)} =_{P/19}$$
$$= x - y - \overline{\underline{(x - z - y)}} =_{Q/26}$$
$$= x - \underline{(y \triangleright x - z - y)} =_{I/31}$$
$$= x - \overline{\underline{(y \triangleright x - z)}} =_{Q/26}$$
$$= x - y - (x - z) =_{P/19}$$
$$= \underline{x - (x - z)} - y =_{@}$$
$$= \overline{x @ z} - y \tag{l/32}$$

$$x @ (y - z) =_{@}$$
$$= x - \underline{(x - (y - z))} =_{Z/}$$
$$= x - \overline{\underline{(x - (x - z) \triangleright x - y)}} =_{Q/26}$$
$$= \underline{x - (x - (x - z))} - (x - y) =_{R/}$$
$$= \overline{x - z} - (x - y) =_{P/19}$$
$$= \underline{x - (x - y)} - z =_{@}$$
$$= \overline{x @ y} - z \tag{m/33}$$

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