Hybrid Automata

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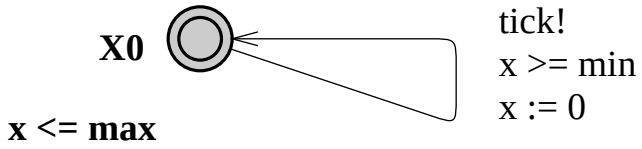
Background

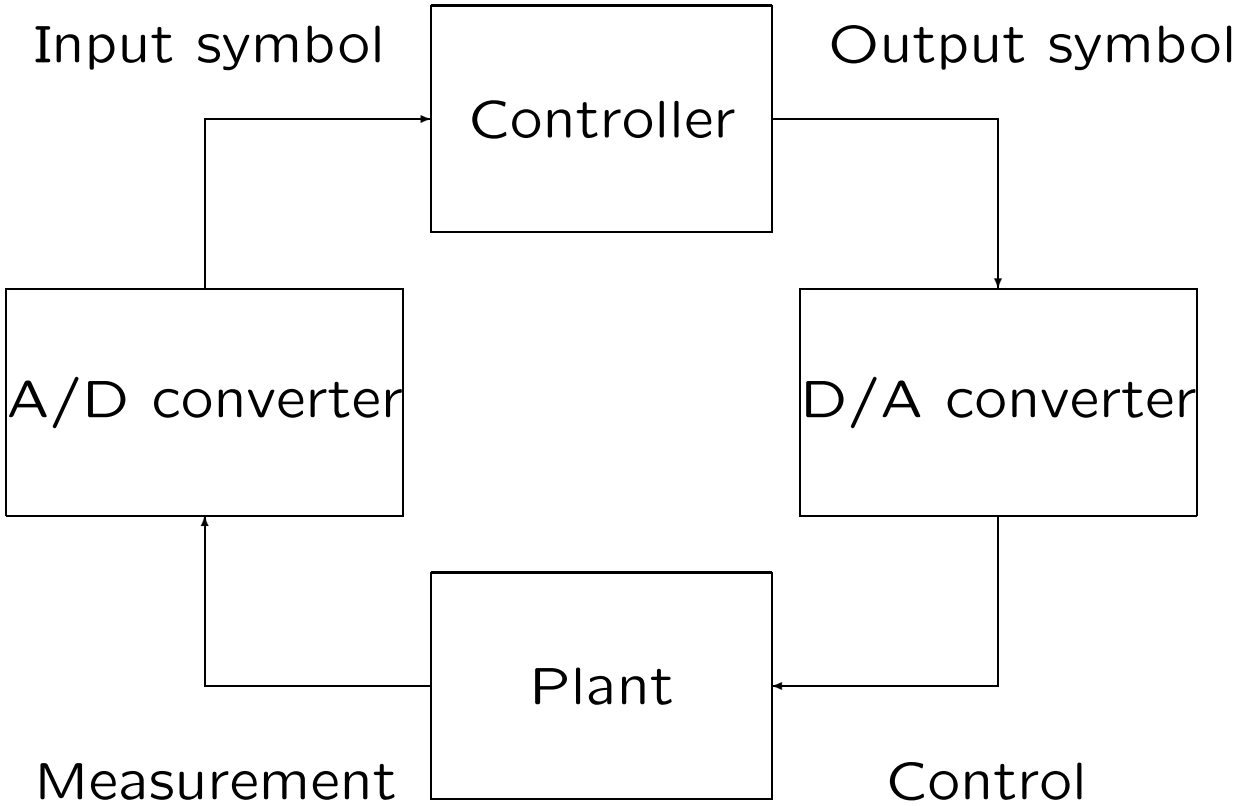
In an Alur-Dill style timed automaton, all clocks proceed with the same rate in each location, i.e. $\dot{x} = 1$ for all clocks x.



We may relax this condition and allow for (continuous) variables that evolve with arbitrary dynamics that may also depend on the location (see e.g. Maler, Manna & Pnueli, 1990).

Such structures are commonly called hybrid automata (HA). Variables may represent a drifting clock, the pressure in a tank, the speed of a car, the temperature in a room, the position of a robot hand, the voltage on a wire, etc.

HAs appropriate modelling formalism to support design and analysis of hybrid control systems:



Two Disciplines

Computer scientists typically try to discretize the plant. Control theorists attempt to make the controller continuous. For many practical applications these approaches don't work; a genuine theory of hybrid systems is required (cf work of Branicky).

Hybrid systems research community aims at developing such a theory.

Model Checking Hybrid Automata

To compute all reachable states of a HA one typically searches exhaustively for new symbolic states until a fixed point is reached. A symbolic state is a pair of a control location and a (typically infinite) set of valuations of the continuous variables.

Unlike for timed automata, for hybrid automata state space exploration will not always terminate. Already for linear hybrid automata ($\dot{x} = n$, for n a natural number) reachability problem is undecidable.

Nevertheless HyTech and PHAVer, model checkers for hybrid automata, have been applied succesfully to several problems.

There are several approaches to compute an over-approximation of the set of reachable states, e.g.,

- orthogonal polyhedra (Dang & Maler)
- projections to lower dimensional polyhedra (Greenstreet & Mitchell)
- ellipsoids (Kurzhanski & Varaiya)
- bounded polyhedra (Chutinan & Krogh, Varaiya, Fehnker)

Also, several restricted classes of hybrid automata have been proposed for which the reachability problem is decidable, e.g.,

- rectangular hybrid automata (Henzinger, Kopke, Puri & Varaiya)
- decrementable timed automata (Fersman, Petterson & Wang)
- linearly priced timed automata (Behrmann et al)

Rather than approximating the set of reachable states, one may analyze an appoximate model of the HA (Henzinger & Ho).

In this lecture, I will focus on the following fundamental issues:

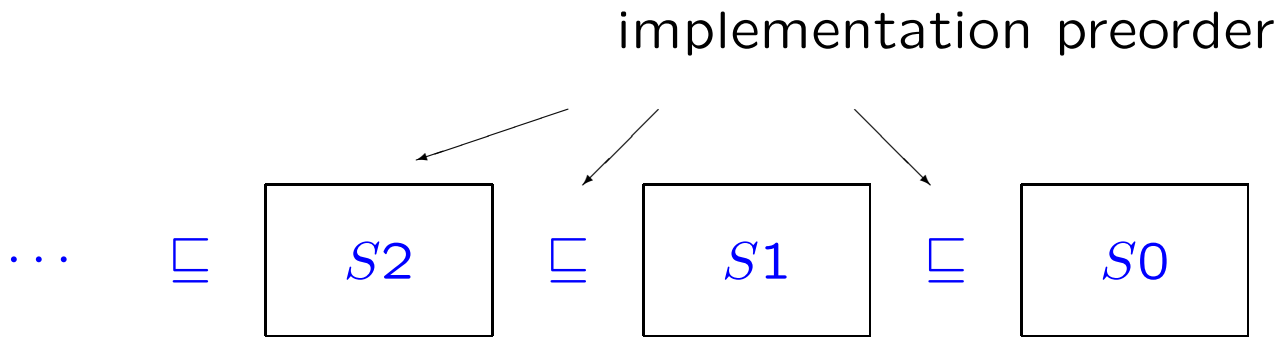
- What is the observable behavior of a HA? What does it mean for one HA to implement another?

- Compositionality

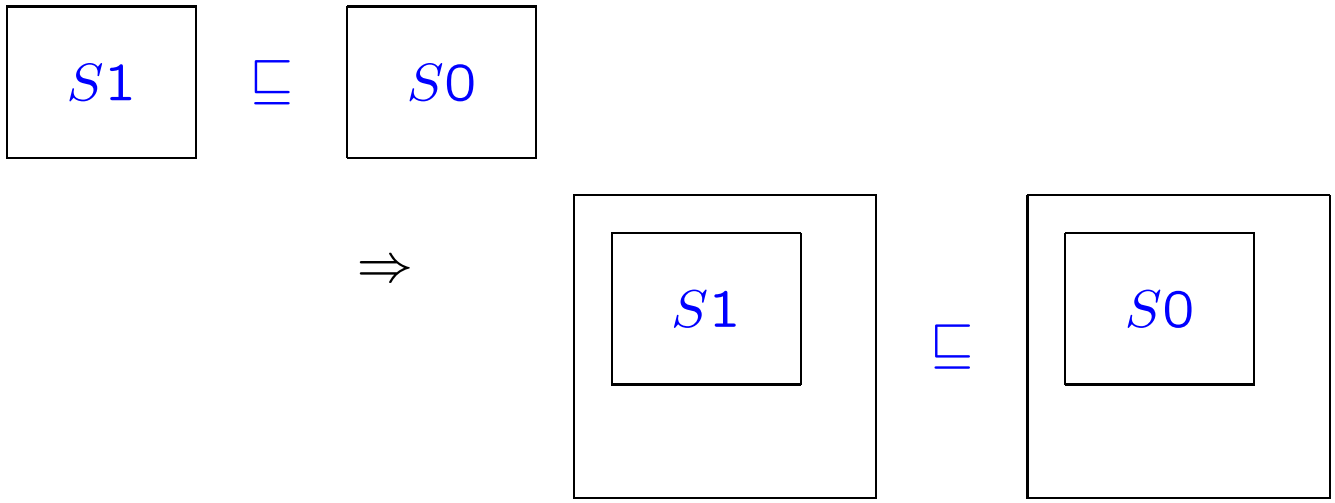
- Receptivity

This is all joint work with Nancy Lynch & Roberto Segala, published in *Information and Computation* in 2003.

Stepwise Refinement



Compositionality



Terminology

The issues that I want to address are best studied at the semantic level. The objects in the semantic world that we define and study will be called hybrid automata, even though this leads to confusion with the syntactic objects with the same name. For the semantic objects, hybrid transition systems probably would have been a better name, just like I/O automata should probably have been called I/O transition systems.

Time

We assume a time axis T, which is a subgroup of $(\mathrm{R}, +)$, the real numbers with addition. We assume that every infinite, monotone, bounded sequence of elements of T has a limit in T.

Examples: the real numbers, the integers, $\{0\}$.

An interval J is a nonempty, convex subset of T.

Types We assume a universal set V of variables.

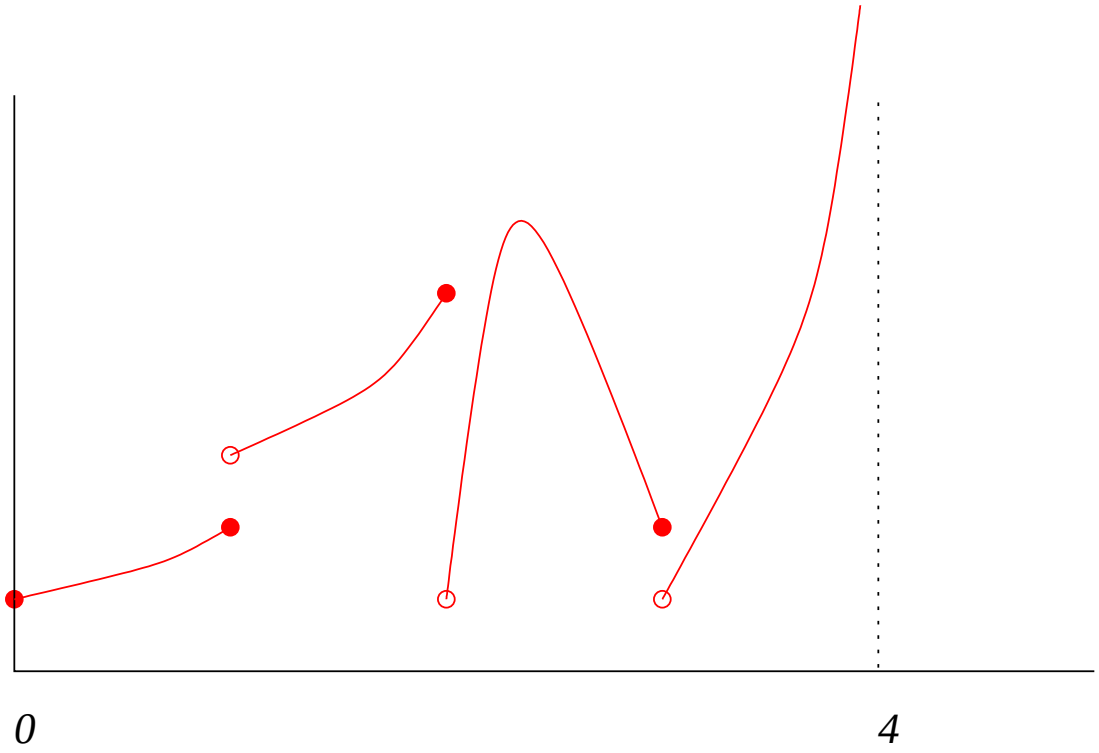
Each variable v has a static type $type(v)$, which is the set of values it may take.

We also assume a dynamic type $dtype(v)$, which is a set of functions from left-closed intervals of T to $type(v)$ that is closed under time shift, subinterval and pasting.

The pasting operations glues together a countable number of functions which all (possibly except for the last one) have a right-closed domain. At borderpoints value of leftmost function is taken.

Examples: (closure of) constant functions, continuous functions, differentiable functions, smooth functions, integrable functions, smooth functions with range $[-1, 1]$...

Example Element of Dynamic Type



Alternatives to pasting closure:
"stuttering" events [LSVW96] or superdense computations [Pnueli94].

Trajectories

Let V be a set of variables and J a left-closed interval of T with left endpoint equal to 0.

A J-trajectory for V is a function $\tau : J \longrightarrow val(V)$, such that for each $v \in V$, $\tau \downarrow v \in dtype(v)$.

Lemma

The set of trajectories for V together with the prefix ordering $\leq$, is an algebraic cpo.

A hybrid automaton (HA) is a tuple $\mathcal{A} = (W, X, Q, \Theta, E, H, D, \mathcal{T})$ with

- W and X disjoint sets of external resp internal variables.
  We call a valuation $\mathbf{x}$ for X a state and write $V \triangleq W \cup X$.
- $Q \subseteq val(X)$ a set of states and $\Theta \subseteq Q$ a nonempty set of start states.
- E and H sets of external resp internal actions.
  We write $A \triangleq E \cup H$ and let $a, a', a_1, a_2, \ldots$ range over A.
- $\mathcal{D} \subseteq Q \times A \times Q$ a set of discrete transitions.
  We write $\mathbf{x} \xrightarrow{a}_{\mathcal{A}} \mathbf{x}'$ for $(\mathbf{x}, a, \mathbf{x}') \in D$.
- A set $\mathcal{T}$ of trajectories for Q

  We require that $\mathcal{T}$ is closed under prefix, suffix and countable concatenation.

Notation

In examples, unless specified otherwise, we take the time domain to be the set of real numbers.

If not specified, we assume the set of states Q equals the set $val(X)$ of all valuations of internal variables.

Notation

We specify sets of trajectories using differential and algebraic equations (or inclusions).

A trajectory satisfies algebraic equation $v = e$ if the constraint on the variables expressed by this equation holds for each point on the trajectory.

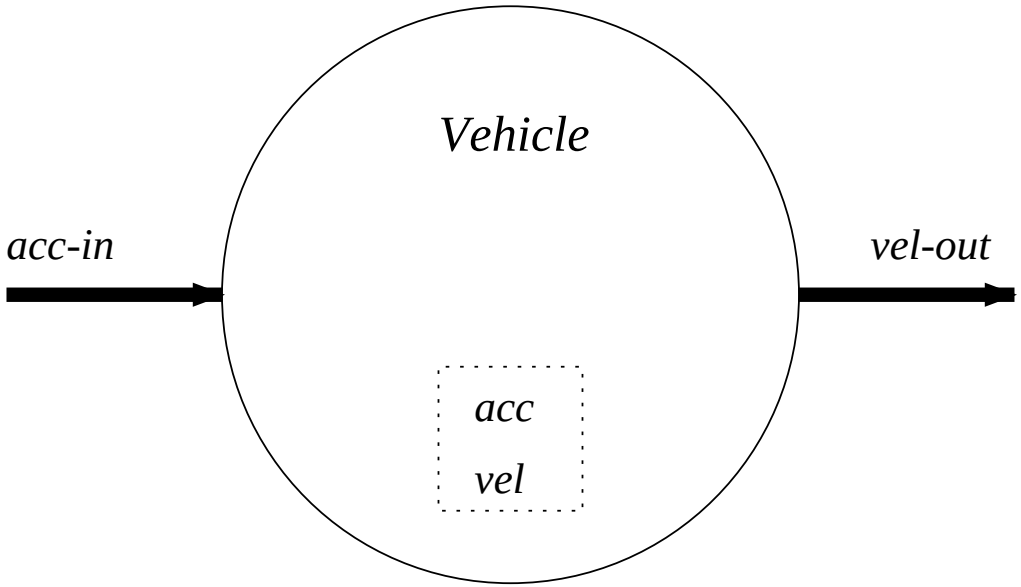
Trajectory τ satisfies differential equation $\dot{v} = e$ if, for every $t \in dom(\tau)$,

$$v(t) = v(0) + \int_0^t e(t')dt'$$

(cf "weak solutions" of Polderman and Willems).

Algebraic/differential inclusions are dealt with similarly.

Example Hybrid automaton $Vehicle$ follows a suggested acceleration approximately, to within an error of $\epsilon \geq 0$.



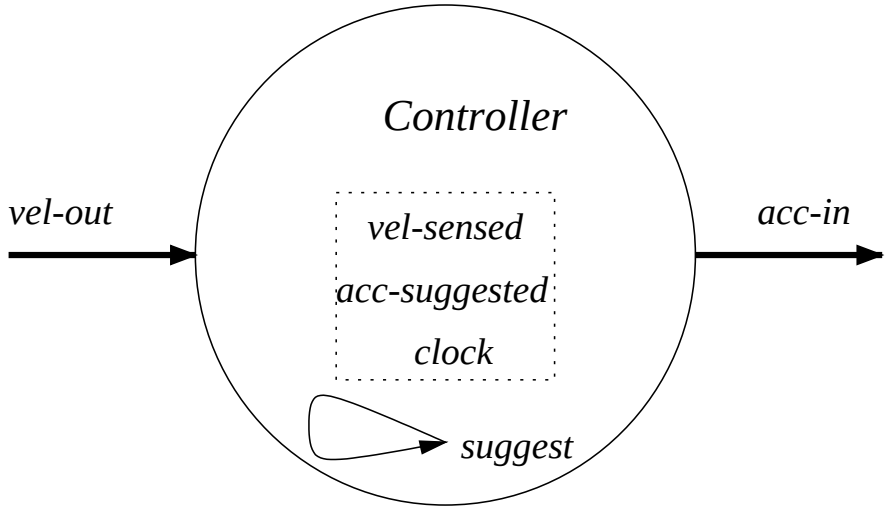
$W = \{acc\text{-}in, vel\text{-}out\}$, $X = \{vel, acc\}$, Θ assigns 0 to both state variables, and E, H and D are empty.

Example (cnt) All variables have type R. The dynamic type of the variables vel, $vel\text{-}out$, and $acc\text{-}in$ is the (pasting closure of the) set of continuous functions. The dynamic type of acc is the set of integrable functions. Set $\mathcal{T}$ consists of all trajectories that satisfy:

$$\begin{aligned} \dot{vel} &= acc \\ acc(t) &\in [acc\text{-}in(t) - \epsilon, acc\text{-}in(t) + \epsilon] \quad \text{for } t > 0 \\ vel\text{-}out &= vel \end{aligned}$$

(No constraints on values input variables in initial state of trajectories.)

Example HA $Controller$ suggests accelerations for a vehicle, with the intention of ensuring that the vehicle's velocity does not exceed a pre-specified velocity vmax.



Q is the set of valuations of X in which $clock \leq$ d, where d is a constant satisfying vmax $\geq \epsilon$ d. Θ assigns 0 to all state variables. $E = \emptyset$ and $H = \{suggest\}$.

Example (cnt)

All variables are of type R. The dynamic types of $vel\text{-}out$, $vel\text{-}sensed$, $acc\text{-}in$, and $clock$ are the (pasting closure of the) set of continuous functions, and $acc\text{-}suggested$ is a discrete variable.

Set D consists of the $suggest$ steps specified by:

$$\begin{aligned} clock &= \mathsf{d} \\ vel\text{-}sensed + (acc\text{-}suggested' + \epsilon)\mathsf{d} &\leq \mathsf{vmax} \\ clock' &= 0 \\ vel\text{-}sensed' &= vel\text{-}sensed \end{aligned}$$

Example (cnt)

Set $\mathcal{T}$ consists of all trajectories that satisfy:

$$\begin{aligned} \dot{acc\text{-}suggested} &= 0 \\ \dot{clock} &= 1 \\ vel\text{-}sensed(t) &= vel\text{-}out(t) \qquad \text{for } t > 0 \\ acc\text{-}in &= acc\text{-}suggested \end{aligned}$$

Executions and Traces

An execution fragment of $\mathcal{A}$ is a sequence $\alpha = \tau_0\, a_1\, \tau_1\, a_2\, \tau_2 \ldots$, where
(1) each τ_i is a trajectory in $\mathcal{T}$, and
(2) if τ_i is not the last trajectory in α then $\tau_i.lstate \overset{a_{i+1}}{\rightarrow} \tau_{i+1}.fstate$.
We say α is an execution if its first state is a start state.

The trace of α, denoted by $trace(\alpha)$, is obtained by
(1) projecting all trajectories of α on the variables in W,
(2) removing the actions in H, and
(3) concatenating all adjacent trajectories.

We define a trace of $\mathcal{A}$ to be the trace of an execution of $\mathcal{A}$.

Implementation

Hybrid automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable if they have the same external interface, that is, if $W_1 = W_2$ and $E_1 = E_2$.
If $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable then $\mathcal{A}_1$ implements A_2, denoted by $\mathcal{A}_1 \leq \mathcal{A}_2$, if the traces of $\mathcal{A}_1$ are included among those of $\mathcal{A}_2$.

Example

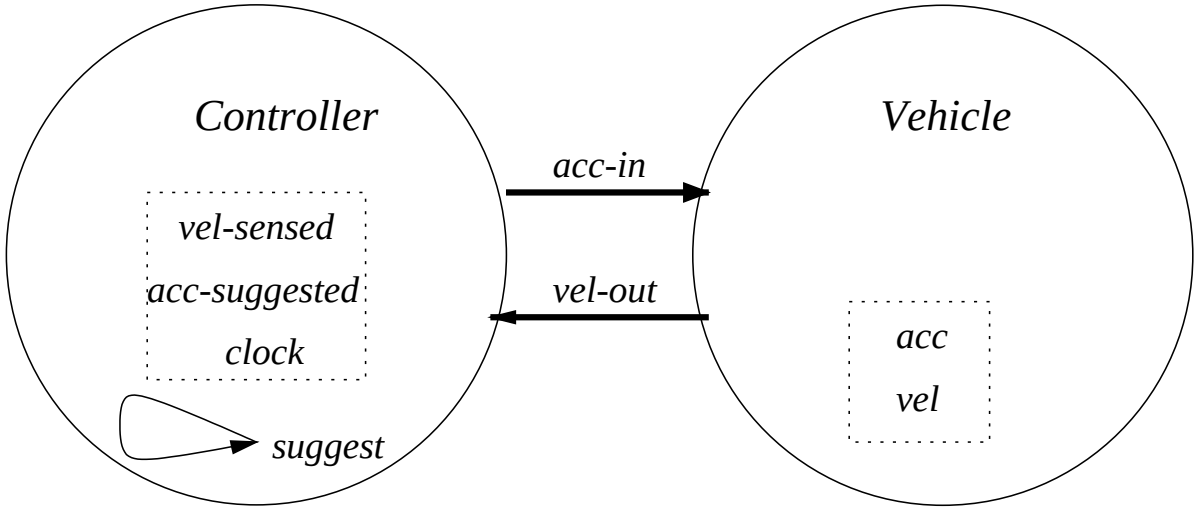
Denote the $Vehicle$ HA by $Vehicle(\epsilon)$, making the uncertainty parameter explicit. Assume $0 \leq \epsilon_1 \leq \epsilon_2$. Then $Vehicle(\epsilon_1) \leq Vehicle(\epsilon_2)$. We can show this by demonstrating that the identity mapping is a simulation relation.

Hybrid automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if $H_1 \cap A_2 = H_2 \cap A_1 = \emptyset$ and $X_1 \cap V_2 = X_2 \cap V_1 = \emptyset$.
In this case the composition $\mathcal{A}_1 \| \mathcal{A}_2$ is the HA $\mathcal{A} = (W, X, Q, \Theta, E, H, D, \mathcal{T})$ where

- $W = W_1 \cup W_2$ and $X = X_1 \cup X_2$.
- $Q = \{\mathbf{x} \in val(X) \mid \mathbf{x} \lceil X_1 \in Q_1 \wedge \mathbf{x} \lceil X_2 \in Q_2\}$.
- $\Theta = \{\mathbf{x} \in Q \mid \mathbf{x} \lceil X_1 \in \Theta_1 \wedge \mathbf{x} \lceil X_2 \in \Theta_2\}$.
- $E = E_1 \cup E_2$ and $H = H_1 \cup H_2$.
- For each $\mathbf{x}, \mathbf{x}' \in Q$ and each $a \in A$, $\mathbf{x} \xrightarrow{a}_{\mathcal{A}} \mathbf{x}'$ iff for $i = 1, 2$, either (1) $a \in A_i$ and $\mathbf{x} \lceil X_i \xrightarrow{a}_i \mathbf{x}' \lceil X_i$, or (2) $a \notin A_i$ and $\mathbf{x} \lceil X_i = \mathbf{x}' \lceil X_i$.
- $\mathcal{T} \subseteq trajs(V)$ is given by $\tau \in \mathcal{T} \Leftrightarrow \tau \downarrow V_1 \in \mathcal{T}_1 \ \wedge \ \tau \downarrow V_2 \in \mathcal{T}_2$.

Example

Consider the $Vehicle$ and $Controller$ automata (for the same ϵ). These two HAs are compatible.



By means of a standard inductive proof one may establish that, for all reachable states of the composed system, $vel \leq$ vmax.

Compositionality

Theorem Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable HAs with $\mathcal{A}_1 \leq \mathcal{A}_2$. Suppose $\mathcal{B}$ is an HA that is compatible with each of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Then $\mathcal{A}_1 \| \mathcal{B}$ and $\mathcal{A}_2 \| \mathcal{B}$ are comparable and $\mathcal{A}_1 \| \mathcal{B} \leq \mathcal{A}_2 \| \mathcal{B}$.

Hiding

We define two hiding operations for hybrid automata, ActHide$(E, \mathcal{A})$ and VarHide$(W, \mathcal{A})$, which hide actions resp variables. Both operations behave well wrt the trace implementation relation.

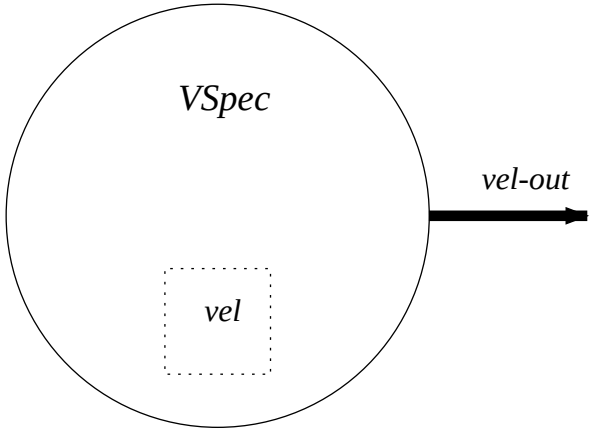
Example In the composition of the $Vehicle$ and $Controller$ HAs, we may hide the $acc\text{-}in$ variable used for communication between the two components. Thus, we define

$$\mathcal{A} \;=\; \text{VarHide}(\{acc\text{-}in\}, Vehicle \| Controller).$$

The only external variable of $\mathcal{A}$ is $vel\text{-}out$.

Correctness

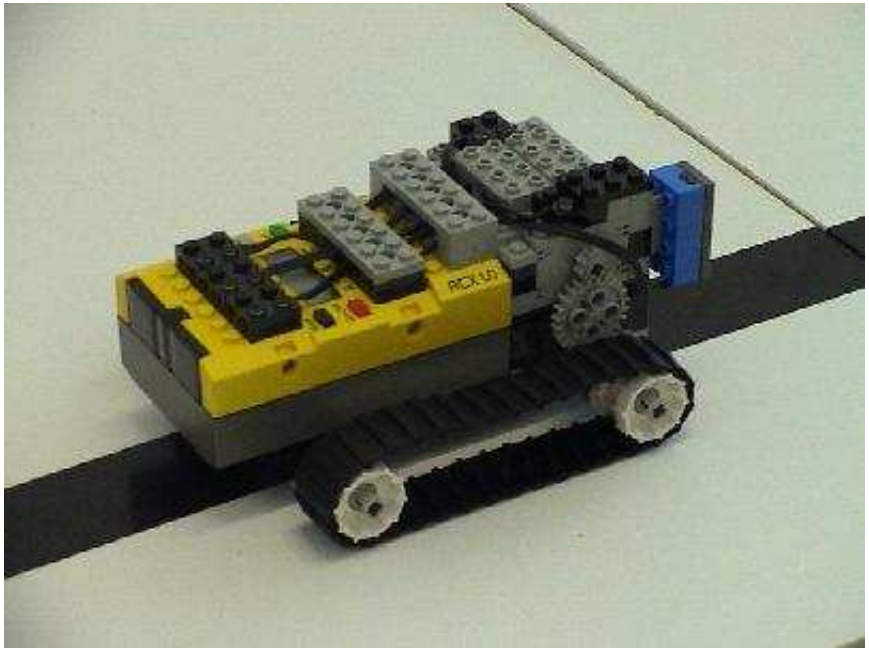
$\mathcal{A}$ implements an abstract specification automaton $VSpec$ that simply represents the constraint that the vehicle's velocity is at most vmax.



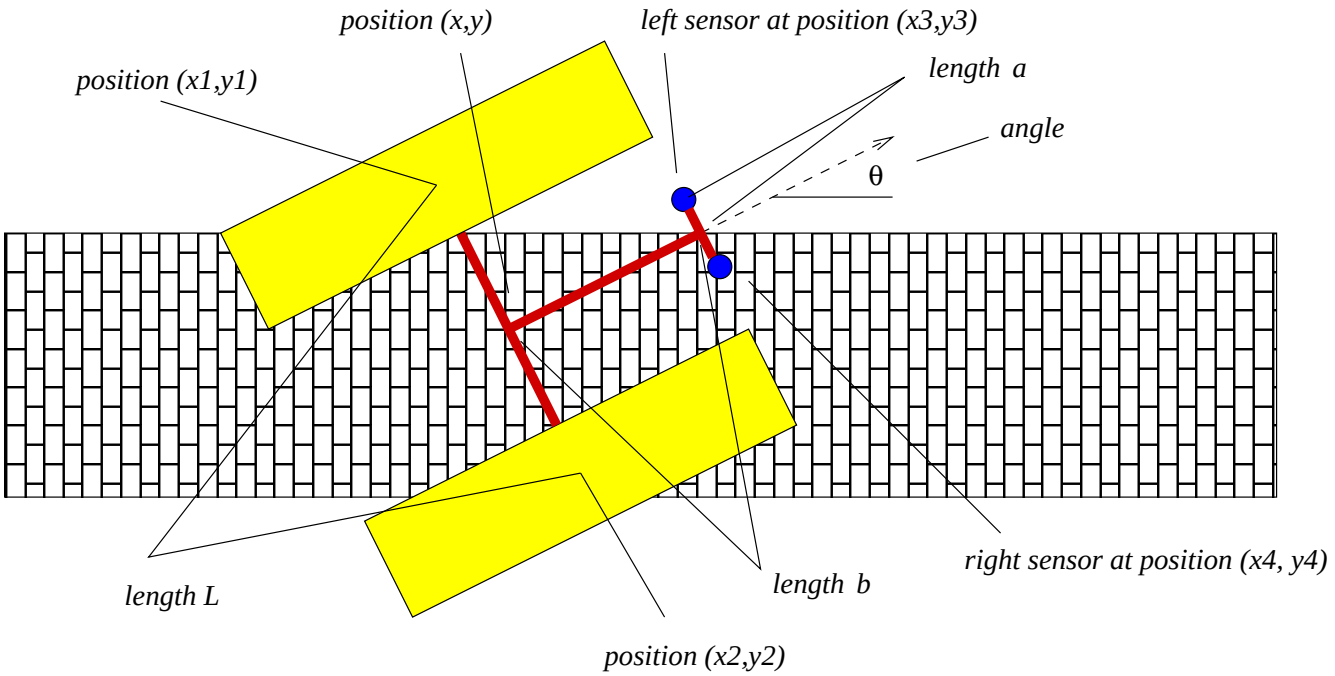
Q is the set of valuations of X in which $vel \leq$ vmax, $\Theta = Q$, $VSpec$ has no actions or discrete transitions. The trajectories of $VSpec$ are those that satisfy $vel\text{-}out = vel$, in each state.

Example: Lego Car

(joint work with Ansgar Fehnker and Miaomiao Zhang)



Operation of LEGO car



As long as sensor sees black background, opposite caterpillar moves forward.

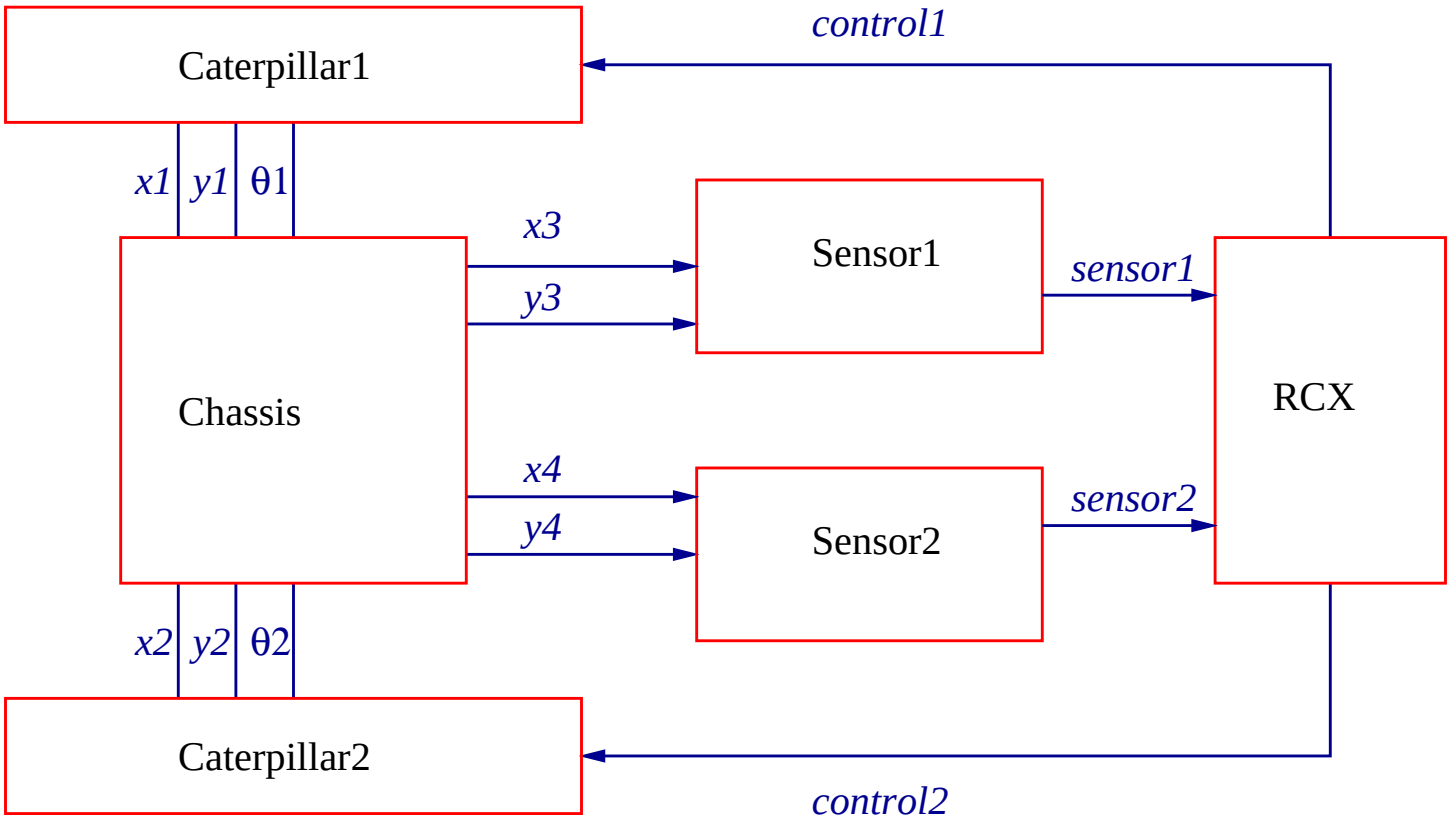
If it sees white background then opposite caterpillar moves backward.

Verification Challenge

If orientation car differs too much from orientation of black tape it may start bumping back and forth between different sides of the tape, and as a result even change direction.

Under which assumptions on the initial orientation can we be sure that the car will always move in a forward direction? (Assuming the tape is infinite)

Network of Hybrid Automata for Lego Car



Chassis

Internal Variables

x, y, θ: differentiable

External Variables

$x1, y1, \theta1, x2, y2, \theta2, x3, y3, x4, y4$: differentiable

Initial States

$\theta \in [-\alpha, \alpha] \wedge y \in [-B, B] \wedge PLS \in [-B, B] \wedge PRS \in [-B, B]$

where $PLS = y + b \sin\theta + a \cos\theta$ and $PRS = y + b \sin\theta - a \cos\theta$

Equations

$$
\begin{aligned}
\theta 1 &= \theta 2 = \theta \\
x1 &= x - \frac{1}{2} L \sin\theta \\
y1 &= y + \frac{1}{2} L \cos\theta \\
x2 &= x + \frac{1}{2} L \sin\theta \\
y2 &= y - \frac{1}{2} L \cos\theta \\
x3 &= x + b\cos\theta - a\sin\theta \\
y3 &= y + b\sin\theta + a\cos\theta \\
x4 &= x + b\cos\theta + a\sin\theta \\
y4 &= y + b\sin\theta - a\cos\theta
\end{aligned}
$$

Caterpillar Treads

External Variables

$x1, y1, \theta 1$: differentiable
control1: Boolean, discrete

Equations

$$\dot{x1} = \textbf{if } \mathit{control1} \textbf{ then } V \cos \theta 1 \textbf{ else } -V \cos \theta 1$$
$$\dot{y1} = \textbf{if } \mathit{control1} \textbf{ then } V \sin \theta 1 \textbf{ else } -V \sin \theta 1$$

Sensors

External Variables

$x3, y3$: differentiable
$sensor1$: discrete, $\{black, white\}$

Equations

$$sensor1 \quad = \quad \textbf{if} \quad y3 \in [-B, B] \quad \textbf{then} \quad black \quad \textbf{else} \quad white$$

RCX

Internal Variables

c: differentiable, $c \leq t_{sample}$
$sample1$, $sample2$: discrete, enumerated type $\{black, white\}$

Initial states

$$c = 0 \ \wedge \ sample1 = sample2 = black$$

External Variables

$sensor1$, $sensor2$: discrete, enumerated type $\{black, white\}$
$control1$, $control2$: discrete Boolean variables

Internal transition

$$c \geq t_{sample} \wedge c' = 0 \wedge sample1' = sensor1 \wedge sample2' = sensor2$$

Variables $sample1$ and $sample2$ remain constant along a trajectory.

Equations

$$
\begin{array}{rcl}
\dot{c} & = & 1 \\
\mathit{control1} & = & \textbf{if} \;\; \mathit{sample2} = \mathit{black} \;\; \textbf{then} \;\; \mathit{true} \;\; \textbf{else} \;\; \mathit{false} \\
\mathit{control2} & = & \textbf{if} \;\; \mathit{sample1} = \mathit{black} \;\; \textbf{then} \;\; \mathit{true} \;\; \textbf{else} \;\; \mathit{false}
\end{array}
$$

Four Modes Depending on Values *control1* and *control2*

$$
\begin{aligned}
control1 \wedge control2 &\Rightarrow \dot{x} = V\cos\theta \wedge \dot{y} = V\sin\theta \wedge \dot{\theta} = 0 \\
control1 \wedge \neg control2 &\Rightarrow \dot{x} = 0 \wedge \dot{y} = 0 \wedge \dot{\theta} = \frac{-2V}{L} \\
\neg control1 \wedge control2 &\Rightarrow \dot{x} = 0 \wedge \dot{y} = 0 \wedge \dot{\theta} = \frac{2V}{L} \\
\neg control1 \wedge \neg control2 &\Rightarrow \dot{x} = -V\cos\theta \wedge \dot{y} = -V\sin\theta \wedge \dot{\theta} = 0
\end{aligned}
$$

Results I

Using a (self written) tool that over approximates the set of reachable states based on bounded polyhedra, Ansgar Fehnker verifed that, assuming that initially the car moves forward with an angle between -45 and 45 degrees:

1. Car always stays on the tape and never moves backward.
2. Right sensor gets never closer to upper bound of tape than 2.1 mm.
3. If car is in forward mode it moves in direction of x-axis with at least 8.9 cm/s (speed of car is 13 cm/s).

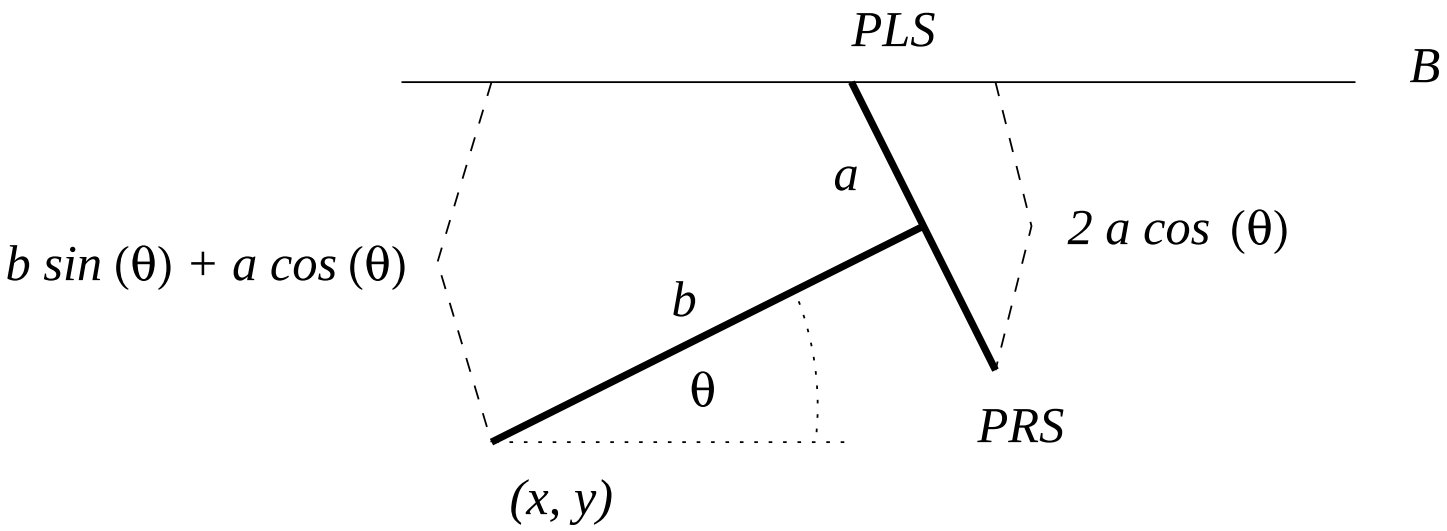
Experiments with the physical car confirm these results.

Results II

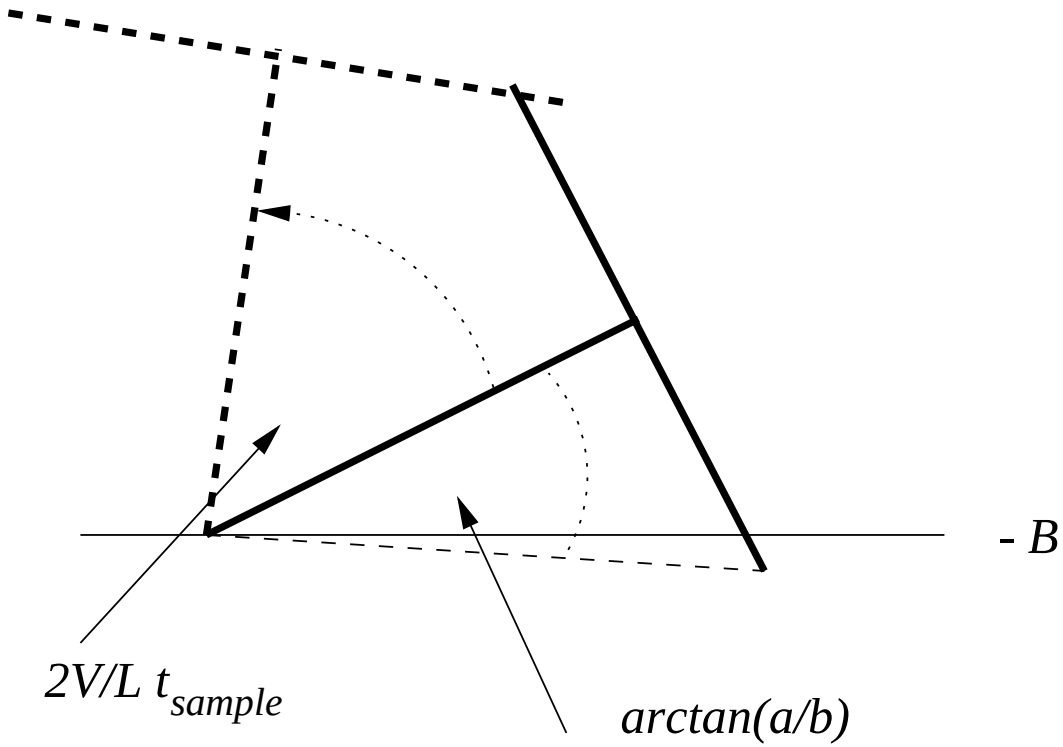
If the following constraints on the parameters hold, the car will never move backward, and infinitely often be in forward mode:

$$
\begin{aligned}
\varphi_1 &= a\cos(\alpha) + b\sin(\alpha) \geq V\sin(\alpha)t_{sample} \\
\varphi_2 &= 2a\cos(\alpha) \geq V\sin(\alpha)t_{sample} \\
\varphi_3 &= \frac{2V}{L}t_{sample} + \arctan(\frac{a}{b}) \leq \alpha \\
\varphi_4 &= a\cos(\frac{V}{L}t_{sample}) + b\sin(\frac{V}{L}t_{sample}) \leq B
\end{aligned}
$$

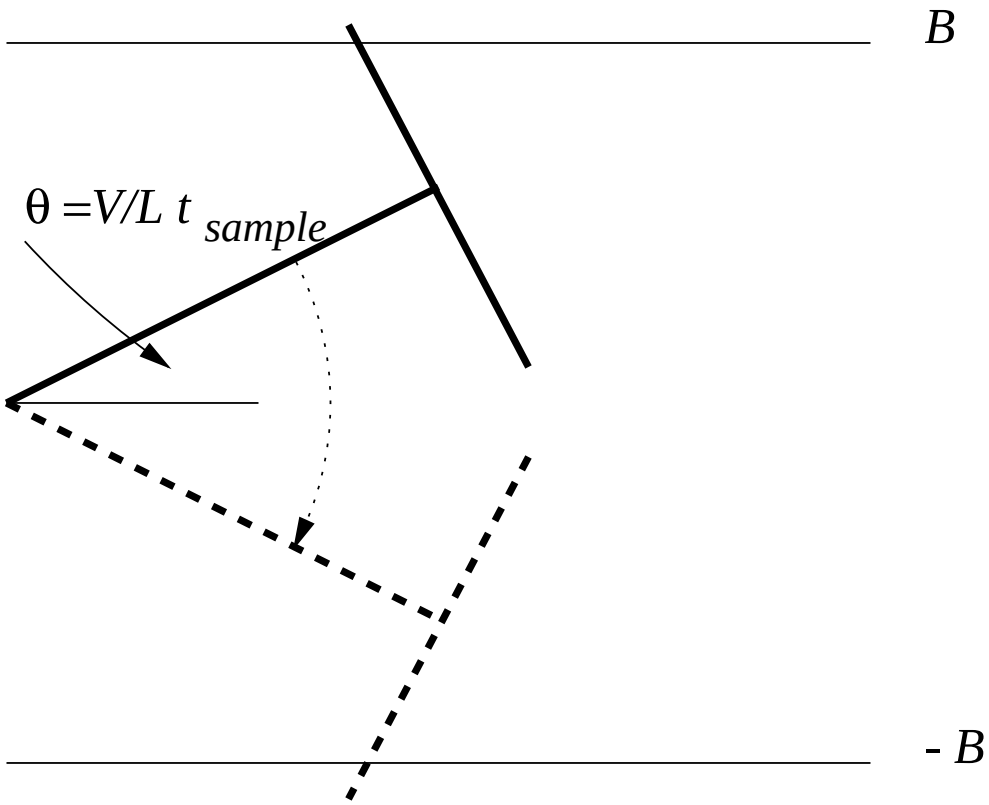
Why are constraints φ_1 and φ_2 needed?



Why is constraint φ_3 needed?



Why is constraint φ_4 needed?



Results III

Extending analysis to include disturbances is easy!!!

I/O Distinction and Input Enabling

Advantages

- helps to avoid mistakes in specifications
- simple semantics in terms of traces
  (no need for failure pairs as in CSP)
- fairness/liveness becomes easier

Disadvantages

- less expressive
  (handshake needed to encode single CSP synchronization)
- process algebra becomes more difficult

Hybrid I/O Automata

A hybrid I/O automaton (HIOA) $\mathcal{A}$ is a tuple $(\mathcal{H}, U, Y, I, O)$ where

- $\mathcal{H} = (W, X, Q, \Theta, E, H, D, \mathcal{T})$ is a hybrid automaton.
- U and Y partition W into input and output variables, resp. Variables in $Z \triangleq X \cup Y$ are called locally controlled; $V \triangleq W \cup X$.
- I and O partition E into input and output actions, resp. Actions in $L \triangleq H \cup O$ are called locally controlled; $A \triangleq E \cup H$.

such that ...

E1 (Input action enabling)
For all $\mathbf{x} \in Q$ and all $a \in I$ there exists $\mathbf{x}' \in Q$ such that $\mathbf{x} \xrightarrow{a} \mathbf{x}'$.

E2 (Input trajectory enabling)
For all $\mathbf{x} \in Q$ and all $\upsilon \in \mathit{trajs}(U)$, there exists $\tau \in \mathcal{T}$ such that $\tau.\mathit{fstate} = \mathbf{x}$, $\tau \downarrow U \leq \upsilon$, and either
1. $\tau \downarrow U = \upsilon$, or
2. τ is closed and some $l \in L$ is enabled in $\tau.\mathit{lstate}$.

A pre-HIOA is a structure as above, except that it need not to satisfy **E1** and **E2**.

Example

Chassis and Caterpillers of LEGO car cannot be viewed as HIOAs

However, their composition is a HIOA

Sensors and RCX are also HIOAs

Composition

Pre-HIOAs $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if $\mathcal{H}_1$ and $\mathcal{H}_2$ are compatible and $Y_1 \cap Y_2 = O_1 \cap O_2 = \emptyset$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible then composition $\mathcal{A}_1 \| \mathcal{A}_2$ is the tuple $\mathcal{A} = (\mathcal{H}, U, Y, I, O)$ where

- $\mathcal{H} = \mathcal{H}_1 \| \mathcal{H}_2$,
- $Y = Y_1 \cup Y_2$,
- $U = (U_1 \cup U_2) - Y$,
- $O = O_1 \cup O_2$, and
- $I = (I_1 \cup I_2) - O$.

Problem The composition of two pre-HIOAs is again a pre-HIOA. However, the composition of two HIOAs is not always a HIOA: the resulting structure not always satisfies **E2**!

Example Suppose $\mathcal{A}_1$ has no discrete steps, input variable v_1, output variable v_2, and as trajectories all functions that satisfy

$$v_2(t) = v_1(t) + 1 \quad \text{for } t > 0$$

Symmetrically, suppose $\mathcal{A}_2$ has no discrete steps, input variable v_2, output variable v_1, and as trajectories all functions that satisfy

$$v_1(t) = v_2(t) + 1 \quad \text{for } t > 0$$

Then the composed system has only point trajectories and does not satisy **E2**.

Theorem If $\mathcal{A}_1$ and $\mathcal{A}_2$ are pre-HIOAs that satisfy **E1**, then the composition $\mathcal{A}_1 \| A_2$ also satisfies **E1**.

Theorem Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two compatible HIOAs such that $U_1 \cap Y_2 = \emptyset$. Then $\mathcal{A}_1 \| \mathcal{A}_2$ is a HIOA.

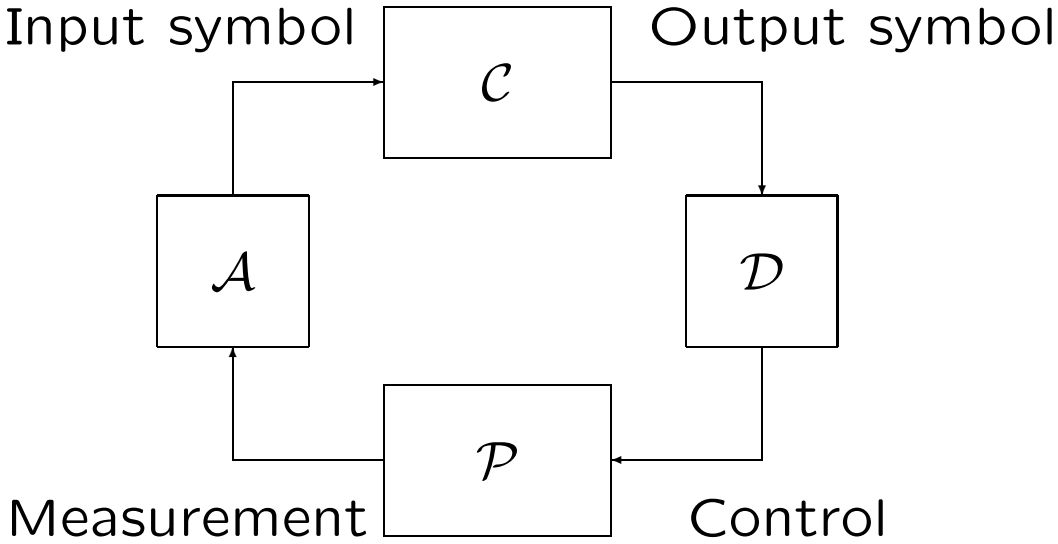
An HIOA is oblivious if it satisfies:

OBL Let $\tau \in \mathcal{T}$ and $\upsilon \in trajs(U)$ such that $dom(\tau) = dom(\upsilon)$. Then there exists $\tau' \in \mathcal{T}$ such that:

1. $\tau' \downarrow U = \upsilon$.
2. $\tau' \downarrow Y = \tau \downarrow Y$.
3. If τ is closed and some locally controlled action is enabled in $\tau.lstate$ then some locally controlled action is enabled in $\tau'.lstate$.

Theorem Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two compatible HIOAs and suppose that $\mathcal{A}_1$ is oblivious. Then $\mathcal{A}_1 \| \mathcal{A}_2$ is a HIOA.

Example: Hybrid Control System



Zeno

An execution fragment is Zeno if it is time-bounded and is either an infinite sequence, or else a finite sequence ending with a trajectory whose domain is right open.

An execution fragment is locally-Zeno if it is Zeno and contains infinitely many locally controlled actions.

A pre-HIOA is progressive if it has no locally-Zeno execution fragments.

Theorem A progressive HIOA is I/O feasible, i.e. able to follow sequence of input trajectories interleaved with input actions.

Theorem The composition of progressive pre-HIOAs is progressive.

Problem

HIOAs involving only upper bounds on timing of events are typically not progressive. Still, we very much like to use such HIOAs in specifications.

Solution

Introduce notion of receptiveness.

Concept has been studied earlier by e.g. Dill and Abadi & Lamport in terms of two-player games. We can use a simpler definition since our model does not involve general liveness properties.

Receptiveness

A strategy for a pre-HIOA $\mathcal{A}$ is an HIOA $\mathcal{A}'$ that differs from $\mathcal{A}$ only in that $D' \subseteq D$ and $\mathcal{T}' \subseteq \mathcal{T}$.

A pre-HIOA is progressive if it has no locally-Zeno execution fragments.

A pre-HIOA is receptive if it has a progressive strategy.

Theorem Every receptive pre-HIOA is I/O feasible.

Theorem Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two compatible receptive HIOAs with progressive strategies $\mathcal{A}'_1$ and $\mathcal{A}'_2$ such that $\mathcal{A}'_1 \| \mathcal{A}'_2$ is an HIOA. Then $\mathcal{A}_1 \| \mathcal{A}_2$ is a receptive HIOA with progressive strategy $\mathcal{A}'_1 \| \mathcal{A}'_2$.

Other Applications of HIOA Framework

- Automated vehicle control systems
  - People movers [Livadas, Lynch][Raytheon]
  - Platoons of cars [Branicky, Dolginova, Lygeros, Lynch]

- Aero/Astro systems
  - Quanser helicopter [Feron, Lynch, Mitra]
  - TCAS collision avoidance system [Livadas, Lygeros, Lynch]
  - Guidance systems [Lynch, Ha] [Draper]

- Algorithms for mobile networks
  - Implementing virtual stationary nodes [Gilbert, Nolte]
  - Motion coordination for mobile robots [Lynch, Mitra, Nolte]

- Control-theory examples
  - Stability analysis for hybrid systems [Mitra, Liberzon]

Conclusions / Future Work

- HIOA model is compositional and supports stepwise refinement.
- Has been applied already tp many examples.
- Examples may come from area of embedded systems but for instance also from biology or psychology.
- Probabilities need to be added.
- Need to incorporate additional analysis methods, e.g. Lyapunov stability analysis and robust control methods.
- Much work required to automate/mechanize these calculations!